

Exact Lemaître-Tolman-Bondi solutions for matter-radiation decoupling

Roberto A. Sussman

Instituto de Ciencias Nucleares UNAM, Apartado Postal 70-458,
México DF, 04510, México

Diego Pavón

Departamento de Física, Universidad Autónoma de Barcelona,
08193 Bellaterra, Spain

Abstract

A new class of exact solutions of Einstein's equations is derived providing a physically plausible hydrodynamic description of cosmological matter in the radiation era of the Universe expansion. The solutions are characterized by the LTB metric with dissipative fluid sources, subjected to the following conditions: (i) the nonequilibrium state variables satisfy the equations of state of a mixture of relativistic and non-relativistic ideal gases, where the internal energy of the latter has been neglected, (ii) the particle numbers of the mixture components are independently conserved, (iii) the viscous stress is consistent with the transport equation and entropy balance law of extended irreversible thermodynamics with the coefficient of shear viscosity provided by kinetic theory. The Jeans mass at decoupling is of the same order of magnitude as that of baryon dominated models (i.e. $M_J \approx 10^{16} M_\odot$).

1 Introduction

The standard approaches to the radiative era of cosmic evolution (the period from the end of cosmic nucleosynthesis to the decoupling of matter and radiation) resort to a FLRW space-time background while the sources of the gravitational field are described either by equilibrium kinetic theory [1], gauge invariant perturbations [2], or some hydrodynamical model [3], which, in general, fail to incorporate a physically plausible description of the matter-radiation interaction since they assume thermodynamical equilibrium throughout. Here we consider the aforesaid sources in the temperature range $10^6 \text{K} \leq T \leq 10^3 \text{K}$, as a nonequilibrium mixture of a non-relativistic fluid (matter) and a extreme relativistic one (radiation); their mutual interaction is modeled by a dissipative shear-stress tensor, and the background space-time is described by the Lemaître-Tolman-Bondi (LTB for short) metric [4]. (The interest for inhomogeneous metrics has revived in the wake of the work of Mustapha *et. al.* [5] who

show that given any spherically symmetric geometry and any set of observations, evolution functions for the observed sources can be found that will make the model in general compatible with observation). The stress-energy tensor reads

$$T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab} \quad (h^{ab} = c^{-2} u^a u^b + g^{ab}),$$

where is assumed that the matter fluid is just “dust” and therefore the hydrostatic pressure is furnished by the radiation only, i.e.

$$\rho \approx mc^2 n^{(m)} + 3n^{(r)} k_B T, \quad p \approx n^{(r)} k_B T$$

In its turn the shear dissipative pressure tensor is governed by

$$\tau \dot{\Pi}_{cd} h_a^c h_b^d + \Pi_{ab} \left[1 + \frac{1}{2} T \eta \left(\frac{\tau}{T \eta} u^c \right)_{;c} \right] + 2\eta \sigma_{ab} = 0, \quad \text{with} \quad \eta_{(rg)} = \frac{4}{5} p^{(r)} \tau,$$

this expression is compatible with relativistic causality and stability, and is supported by kinetic theory, statistical fluctuation theory, and information theory [6]. The entropy per particle takes the form

$$s = s^{(e)} + \frac{\alpha}{nT} \Pi_{ab} \Pi^{ab}, \quad \Rightarrow \quad (s n u^a)_{;a} \geq 0,$$

with

$$\alpha_{(rg)} = -\frac{\tau}{2\eta_{(rg)}} = -\frac{5}{8p^{(r)}},$$

and τ is the relaxation time for the shear-stress. Further, since the number of particles of each fluid is independently conserved, we have $(n^{(m)} u^a)_{;a} = (n^{(r)} u^a)_{;a} = 0$.

2 Exact Solution and Density Contrasts

The LTB metric can be written as

$$ds^2 = -c^2 dt^2 + \frac{Y'^2}{1-F} dr^2 + Y^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \quad Y = Y(t, r), \quad F = F(r).$$

From this metric element the expansion and shear read

$$\Theta = \frac{\dot{Y}'}{Y'} + \frac{2\dot{Y}}{Y}, \quad \sigma \equiv \frac{1}{3} \left(\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y'} \right), \quad \sigma^a{}_b = \mathbf{diag}[0, -2\sigma, \sigma, \sigma],$$

respectively; and the most general form for the shear-stress reduces to $\Pi^a{}_b = \mathbf{diag}[0, -2P, P, P]$, with $P = P(t, r)$. The non-trivial Einstein's field equations are

$$\begin{aligned} \kappa \rho &= -\frac{[Y(\dot{Y}^2 + Fc^2)]'}{Y^2 Y'} = -G^t{}_t \\ \kappa p &= -\frac{[Y(\dot{Y}^2 + Fc^2) + 2Y^2 \ddot{Y}]'}{3Y^2 Y'} = \frac{1}{3} (2G^\theta{}_\theta + G^r{}_r), \end{aligned}$$

$$\kappa P = \frac{Y}{6Y'} \left[\frac{Y (\dot{Y}^2 + F c^2) + 2Y^2 \ddot{Y}}{Y^3} \right]' = \frac{1}{3} (G^\theta_\theta - G^r_r),$$

where the prime and overdot mean partial derivatives with respect to r and t , respectively. In the case of flat spatial sections ($F = 0$), integration of these equations yields the following exact solution [4]

$$\frac{3}{2} \sqrt{\mu} (t - t_i) = \sqrt{y + \epsilon} (y - 2\epsilon) - \sqrt{1 + \epsilon} (1 - 2\epsilon),$$

where we have used the definitions

$$\mu \equiv \frac{\kappa M}{Y_i^3}, \quad \epsilon \equiv \frac{W}{M}, \quad y \equiv \frac{Y}{Y_i},$$

with

$$M = \int \rho_i^{(m)} Y_i^2 Y_i' dr, \quad \rho_i^{(m)} \equiv m c^2 n_i^{(m)}, \quad W = \int \rho_i^{(r)} Y_i^2 Y_i' dr, \quad \rho_i^{(r)} \equiv 3 n_i^{(r)} k_B T_i,$$

so that $\rho_i^{(m)}$, $\rho_i^{(r)}$ define the initial densities of the non-relativistic and relativistic components of the mixture, respectively.

At this point it is expedient to introduce the density contrast parameters

$$\rho_i^{(m)} = \langle \rho_i^{(m)} \rangle [1 + \Delta_i^{(m)}], \quad \rho_i^{(r)} = \langle \rho_i^{(r)} \rangle [1 + \Delta_i^{(r)}],$$

as well as the ancillary functions

$$\Gamma \equiv \frac{Y'/Y}{Y_i'/Y_i} \quad \Psi \equiv 1 + \frac{(1 - \Gamma)}{3(1 + \Delta_i^{(r)})}, \quad \Phi \equiv 1 + \frac{(1 - 4\Gamma)}{3(1 + \Delta_i^{(r)})},$$

the latter are linked to the density contrasts by

$$\begin{aligned} \Gamma &= 1 + 3A\Delta_i^{(m)} + 3B\Delta_i^{(r)}, \\ \Psi &= 1 - \frac{A\Delta_i^{(m)} + B\Delta_i^{(r)}}{1 + \Delta_i^{(r)}}, \\ \Phi &= \frac{-4A\Delta_i^{(m)} + (1 - 4B)\Delta_i^{(r)}}{1 + \Delta_i^{(r)}}, \end{aligned}$$

where the quantities A and B are known functions of y . Given a set of initial conditions specified by ϵ , $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, on some initial hypersurface, the forms of the state and geometric variables as functions of y and the chosen initial conditions are fully determined. However, for the solutions to be physically meaningful they must comply with the following set of physical restrictions,

$$\dot{s} = \frac{15k_B}{16\tau} \left(\frac{\Phi}{\Psi} \right)^2 > 0, \quad \Gamma > 0, \quad \Psi > 0, \quad \sigma\Phi < 0,$$

$$\dot{\tau} > 0, \quad \frac{\ddot{s}}{\dot{s}} = \frac{2\sigma\Gamma}{3\Psi\Phi} \frac{\langle \rho_i^{(r)} \rangle}{\rho_i^{(r)}} \left[1 + \frac{\langle \rho_i^{(r)} \rangle}{3\rho_i^{(r)}} \right] - \frac{\dot{\tau}}{\tau} < 0,$$

$$\tau < 3/\Theta \quad \text{before decoupling}, \quad \tau > 3/\Theta \quad \text{after decoupling}.$$

3 Initial Perturbations and Jeans Mass

Assuming that the perturbations on the initial hypersurface are non-adiabatic (i.e. $\Delta_i^{(s)} \equiv \frac{3}{4}\Delta_i^{(r)} - \Delta_i^{(m)} \neq 0$) and that $T_i \approx 10^6$ Kelvin, the set of values associated to the matter-radiation decoupling is obtained from solving numerically the equation [4]

$$T_D \approx 4 \times 10^3 = \frac{10^6}{y_D} \Psi(y_D, \Delta_i^{(s)}, \Delta_i^{(r)}), \quad (T = \frac{T_i}{y} \Psi),$$

which results in $y_D \approx 10^{2.4}$.

The temperature anisotropy of the CMR ($[\delta T/T]_D \approx 10^{-5}$) constrains the maximal values of $\Delta_i^{(m)}$, $\Delta_i^{(r)}$ to about 10^{-4} and the corresponding variation range of $|\Delta_i^{(s)}|$ is $|\Delta_i^{(s)}| < 10^{-8}$, and so compatibility with acceptable values of T_D and the CMR anisotropy implies $|\Delta_i^{(s)}| \approx |\Delta_i^{(r)}|^2 \ll |\Delta_i^{(r)}|$.

For non-adiabatic perturbations the Jeans mass is given by

$$M_J = \frac{4\pi}{3} m n^{(m)} \left[\frac{8\pi^2 C_s^2}{\kappa(\rho + p)} \right]^{3/2} = \frac{4\pi}{3} \frac{c^4 \chi_0 \Gamma^{1/2}}{\sqrt{\rho_i^{(r)}}} \left[\frac{\pi y^2 \Psi}{3G \left(\Psi + \frac{3}{4} \chi_i y \right)^2} \right]^{3/2},$$

where

$$C_s^2 = \frac{c^2}{3} \left[1 + \frac{3\rho^{(m)}}{4\rho^{(r)}} \right]^{-1}, \quad \chi_i = \rho_i^m / \rho_i^r.$$

Assuming $y = y_D \approx 10^{2.4}$, $\epsilon \approx 1/\chi_i \approx 10^3$ and $\rho_i^{(r)} \approx a_B T_i^4 \approx 7.5 \times 10^9 \text{ ergs/cm}^3$, yields $M_J \approx 10^{49} \text{ gm}$, or approximately 10^{16} solar masses. This value coincides with the Jeans mass obtained for baryon dominated perturbative models as decoupling is approached in the radiative era.

4 Conclusions and Outlook

The model presented here is a self consistent hydrodynamical approach to matter-radiation mixtures that: (a) is based on inhomogeneous exact solutions of Einstein's field equations, and (b) is thermodynamically consistent. It has, however, the limitations of not incorporating heat currents (the LTB models do not allow for this[7]) nor bulk dissipative stresses. Nonetheless, we believe this model may become a useful theoretical tool in the study of cosmological matter sources, providing a needed alternative and complement to the usual perturbative or numerical approaches.

The solutions have an enormous potential as models in applications of astrophysical and cosmological interest:

Structure formation in the acoustic phase. There is a large body of literature on the study of acoustic perturbations in relation to the Jeans mass of surviving cosmological condensations. Equations of state of the type “dust plus radiation” are oftenly suggested in this context [2]. Since practically all work on this topic has been carried on with perturbations on a FLRW background, the exact solutions presented here may be viewed as an alternative treatment for this problem.

Comparison with perturbation theory. Our model is based on exact solutions of Einstein’s field equations, but their initial conditions and evolution can be adapted for a description of “exact spherical perturbations” on a FLRW background. It would be extremely interesting, not only to compare the results of this approach with those of a perturbative treatment, but to provide a physically plausible theoretical framework to examine carefully how much information is lost in the non-linear regime that falls beyond the scope of linear perturbations. We have considered only the case $F = 0$, thus restricting the evolution to the “growing mode” since all fluid layers expand monotonously. The study of the more general case, where $F(r)$ is an arbitrary function that could change sign, would allow a comparison with perturbations that include also a “decay-mode” related to condensation and collapse of cosmological inhomogeneities.

Inhomogeneity and irreversibility in primordial density perturbations. The initial conditions of the models with $\Delta_i^{(s)} \neq 0$ are set for a hypersurface with temperature $T_i \approx 10^6\text{K}$. These initial conditions can be considered the end product of processes characteristic of previous cosmological history, and so the estimated value $|\Delta_i^{(s)}| \approx 10^{-8}$, related to the spatial variation of photon entropy fluctuations, can be used as a constraint on the effects of inhomogeneity on primordial entropy fluctuations that might be predicted by inflationary models at earlier cosmic time. Also, the deviation from equilibrium in the initial hypersurface (proportional to $[\Delta_i^{(r)}]^2 \approx |\Delta_i^{(s)}|$) might be helpful to understand the irreversibility associated with the physical processes involved in the generation of primordial perturbations [8].

Acknowledgments

This work has been partially supported by the Spanish Ministry of Education and the National University of Mexico (UNAM), under grants PB94-0718 and DGAPA-IN-122498, respectively.

References

- [1] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, New York, 1990); T. Padmanabhan, *Formation of Structures in the Universe* (CUP, Cambridge, 1995); J. Bernstein, *Kinetic Theory in the Expanding Universe* (CUP, Cambridge, 1988).
- [2] G. Börner, *The Early Universe, Facts and Fictions* (Springer, Berlin, 1988); T. Padmanabhan, *Formation of Structures in the Universe* (CUP, Cambridge, 1995); P. J. E. Peebles, *The Large Scale Structure of the Universe* (Princeton University Press, Princeton, 1980).
- [3] A. A. Coley and B. O. J. Tupper, J. Math. Phys., **27**, 406 (1986); R. A. Sussman, Class. Quantum Grav., **9**, 1981 (1992); V. Méndez and D. Pavón, MNRAS, **782**, 753 (1996).
- [4] R. A. Sussman and D. Pavón, Phys. Rev. D, **60**, 104023 (1999).
- [5] N. Mustapha, C. Hellaby, and G. F. R. Ellis, MNRAS, **292**, 817 (1997).
- [6] W. Israel and J. Stewart, Ann. Phys. (NY), **118**, 341 (1979); W. A. Hiscock and L. Lindblom, Phys. Rev. D, **35**, 3723 (1987); D. Jou, J. Casas-Vázquez and G. Lebon, *Extended Irreversible Thermodynamics* 2nd ed. (Springer, Berlin, 1996).
- [7] H. Bondi, MNRAS, **107**, 410 (1947).
- [8] A. Linde, *Particle Physics and Inflationary Cosmology*, Fifth printing (Harwood, New York, 1996).